

On the Monodromy and Galois Group of Conics Lying on Heisenberg Invariant Quartic K3 Surfaces

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Abstract

In [Ekl10], Eklund showed that a general $(\mathbb{Z}/2\mathbb{Z})^4$ -invariant quartic K3 surface contains at least 320 conics. In this paper we analyse the field of definition of those conics as well as their Monodromy group. As a result, we prove that the moduli space $(\mathbb{Z}/2\mathbb{Z})^4$ -invariant quartic K3 surface with a marked conic has 10 irreducible components.

1 Introduction

Consider the $(\mathbb{Z}/2\mathbb{Z})^4$ subgroup of $\text{Aut}(\mathbb{P}_{\mathbb{Q}}^3)$ generated by the four transformations

$$[x : y : z : w] \mapsto [y : x : w : z], [z : w : x : y], [x : y : -z : -w], [x : -y : z : -w].$$

The family of all quartic surfaces in $\mathbb{P}_{\mathbb{Q}}^3$ which are invariant under these transformations is known to be parameterised by \mathbb{P}^4 and has been studied extensively by [BN94, Ekl10]. In his paper, [Ekl10], Eklund shows that a general such quartic surface contains at least 320 conics. This paper is lead by the question, if such a surface is defined over a number field K , what is the smallest extension for the conics on it to be defined? Upon answering this question we look into the Monodromy group of the conics and link it back to the Galois group of the field extension.

As a result of this paper, we deduce that the moduli space of pairs (X, C) , where X is an Heisenberg-invariant quartic K3 surfaces and C a conic on X , has 10 irreducible components. We contrast this with universal Severi varieties of K3 surfaces. Let X be a primitive K3 surface of genus g , and L an ample primitive line bundle on X such that $L^2 = 2g - 2$ (so $g > 1$). A Severi variety of X is $V_{k,h}(X) = \{C \in |kL| : C \text{ is irreducible nodal and } g(C) = h\}$. The universal variety $\mathcal{V}_{k,h}^g$ can be considered as the moduli space of pairs (X, C) , where X is a primitive K3 surface of genus g , and $C \in V_{k,h}(X)$. It is conjecture that all universal Severi varieties $\mathcal{V}_{k,h}^g$ are irreducible. This conjecture has been proven by Ciliberto and Dedieu for $3 \leq g \leq 11$, $g \neq 10$ in [CD12]. Kemeny showed that $\overline{\mathcal{V}}_{0,1}^g$ is connected for $g > 2$ in [Kem13], if it can be proven that $\overline{\mathcal{V}}_{0,1}^g$ is smooth then it is irreducible.

In Section 2 we set the notations and review known results that we will use in the rest of the paper. In Section 3 we start with the K3 surface parameterised by a point $p \in \mathbb{P}^4$ and find the explicit equations, in terms of p , of the lines and conics lying said K3 surface. We use this to construct the field extension and show that, in general, the Galois group of the field extension is C_2^{10} . We then look at the Monodromy group of the conics in Section 4. In particular, we find generators of the group and how it acts on the 320 conics as well as showing it is isomorphic to the Galois group.

Note. Throughout this paper, most calculations and equation rearrangements were done using the computer algebra package Magma [BCP97, V2.21-4].

2 Background

We will be studying the following scheme

$$\mathcal{X} := \{A(x^4 + y^4 + z^4 + w^4) + Bxyzw + C(x^2y^2 + z^2w^2) + D(x^2z^2 + y^2w^2) + E(x^2w^2 + y^2z^2) = 0\} \subset \mathbb{P}_{[x:y:z:w]}^3 \times \mathbb{P}_{[A:B:C:D:E]}^4$$

defined over $\overline{\mathbb{Q}}$. This can be viewed as a family of quartic surfaces over \mathbb{P}^3 parameterised by points $[A, B, C, D, E]$ in \mathbb{P}^4 .

Notation. We will use X_p and $[A, B, C, D, E]$ to denote the surface parametrised by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$.

Note. In the case the surface X_p is smooth, then it is a K3 surface.

Consider the group Ω acting on $\mathbb{P}^3 \times \mathbb{P}^4$ generated by the following five elements

$$[x, y, z, w, A, B, C, D, E] \mapsto \begin{cases} [x, y, z, -w, A, -B, C, D, E] & \phi_1 \\ [x, y, w, z, A, B, C, E, D] & \phi_2 \\ [x, z, y, w, A, B, D, C, E] & \phi_3 \\ [x, y, iz, iw, A, -B, C, -D, -E] & \phi_4 \\ [x - y, x + y, z - w, z + w, 2A + C, 8(D - E), 12A - 2C, B + 2D + 2E, -B + 2D + 2E] & \phi_5 \end{cases}.$$

This group fixes \mathcal{X} and hence is a subgroup of its automorphism group. While this group is rather large with order $2^4 \cdot 6!$, we can pick out the following normal subgroup generated by

- $\gamma_1 := \phi_3 \phi_4^2 \phi_3 \phi_5^2$,
- $\gamma_2 := \phi_4^2 \phi_3 \phi_5^2 \phi_3$,
- $\gamma_3 := \phi_4^2$,
- $\gamma_4 := \phi_3 \phi_4^2 \phi_3$,

which we denote by Γ . The group Γ consists of all elements of Ω which fix $\mathbb{P}_{[A:B:C:D:E]}^4$ in $\mathbb{P}^3 \times \mathbb{P}^4$. In particular upon picking a point $p \in \mathbb{P}^4$ we have that Γ is a subgroup of $\text{Aut}(X_p)$ (when projecting the elements of Γ onto the $\mathbb{P}_{[x:y:z:w]}^3$ component). Explicitly, when regarding Γ as acting on \mathbb{P}^3 , we have that its generators are

$$[x, y, z, w] \mapsto \begin{cases} [y, x, w, z] & \gamma_1 \\ [z, w, x, y] & \gamma_2 \\ [x, y, -z, -w] & \gamma_3 \\ [x, -y, z, -w] & \gamma_4 \end{cases}.$$

From this we know that $\Gamma \cong C_2^4$. We also note that $\Omega/\Gamma \cong S_6$ (but $\Omega \not\cong C_2^4 \times S_6$ in particular because Ω has trivial centre). We also make a note of the points of \mathbb{P}^3 which are fixed by $\gamma \in \Gamma \setminus \{\text{id}\}$ (restricted to \mathbb{P}^3). Each such γ has two skew line L, \overline{L} of fixed points which are given by its $(+1)$ and (-1) eigenspace, respectively its $(+i)$ and $(-i)$ eigenspace. The lines are given in Table 1 (more information is given which will be explained after Proposition 2.4). Note that every pair of lines is also fixed by any $\gamma \in \Gamma$ (on top of containing the fixed points of a particular γ).

Notation. We shall denote by \mathcal{L} the union of the 15 pairs of lines.

We now consider the cases when X_p is not a smooth surface using the following proposition taken from [Ekl10, Prop 2.1].

Proposition 2.1. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$. The surface X_p is singular if and only if*

$$A \cdot \Delta \cdot q_{+C} \cdot q_{-C} \cdot q_{+D} \cdot q_{-D} \cdot q_{+E} \cdot q_{-E} \cdot p_{+0} \cdot p_{+1} \cdot p_{+2} \cdot p_{+3} \cdot p_{-0} \cdot p_{-1} \cdot p_{-2} \cdot p_{-3} = 0,$$

where:

$$\Delta = 16A^3 + AB^2 - 4A(C^2 + D^2 + E^2) + 4CDE \quad (1)$$

	L_i	\bar{L}_i	Segre Plane
γ_1	$[s : s : t : t]$	$[s : -s : t : -t]$	$q_{+C} = p_{+0} = p_{-1} = 0$
γ_2	$[s : t : s : t]$	$[s : t : -s : -t]$	$q_{+D} = p_{+0} = p_{-2} = 0$
$\gamma_1\gamma_2$	$[s : t : t : s]$	$[s : t : -t : -s]$	$q_{+E} = p_{+0} = p_{-3} = 0$
γ_3	$[s : t : 0 : 0]$	$[0 : 0 : s : t]$	$A = q_{+C} = q_{-C} = 0$
$\gamma_1\gamma_3$	$[s : -s : t : t]$	$[s : s : t : -t]$	$q_{-C} = p_{-0} = p_{+1} = 0$
$\gamma_2\gamma_3$	$[s : t : is : it]$	$[s : t : -is : -it]$	$q_{-D} = p_{-1} = p_{+3} = 0$
$\gamma_1\gamma_2\gamma_3$	$[s : t : it : is]$	$[s : t : -it : -is]$	$q_{-E} = p_{-1} = p_{+2} = 0$
γ_4	$[s : 0 : t : 0]$	$[0 : s : 0 : t]$	$A = q_{+D} = q_{-D} = 0$
$\gamma_1\gamma_4$	$[s : is : t : it]$	$[s : -is : t : -it]$	$q_{-C} = p_{-2} = p_{+3} = 0$
$\gamma_2\gamma_4$	$[s : t : -s : t]$	$[s : t : s : -t]$	$q_{+D} = p_{-0} = p_{+2} = 0$
$\gamma_1\gamma_2\gamma_4$	$[s : t : it : -is]$	$[s : t : -it : is]$	$q_{-E} = p_{+1} = p_{-2} = 0$
$\gamma_3\gamma_4$	$[s : 0 : 0 : t]$	$[0 : s : t : 0]$	$A = q_{+E} = q_{-E} = 0$
$\gamma_1\gamma_3\gamma_4$	$[s : -is : t : it]$	$[s : is : t : -it]$	$q_{-C} = p_{+2} = p_{-3} = 0$
$\gamma_2\gamma_3\gamma_4$	$[s : t : -is : it]$	$[s : t : is : -it]$	$q_{-D} = p_{+1} = p_{-3} = 0$
$\gamma_1\gamma_2\gamma_3\gamma_4$	$[s : t : t : -s]$	$[s : t : -t : s]$	$q_{+W} = p_{-0} = p_{+3} = 0$

Table 1: List of Invariant Lines

$$\begin{array}{lll}
q_{+C} = 2A + C & q_{+D} = 2A + D & q_{+E} = 2A + E \\
q_{-C} = 2A - C & q_{-D} = 2A - D & q_{-E} = 2A - E \\
p_{+0} = 4A + B + 2C + 2D + 2E & & p_{-0} = 4A - B + 2C + 2D + 2E \\
p_{+1} = 4A + B + 2C - 2D - 2E & & p_{-1} = 4A - B + 2C - 2D - 2E \\
p_{+2} = 4A + B - 2C + 2D - 2E & & p_{-2} = 4A - B - 2C + 2D - 2E \\
p_{+3} = 4A + B - 2C - 2D + 2E & & p_{-3} = 4A - B - 2C - 2D + 2E.
\end{array}$$

Definition 2.2. The surface $S_3 = \{\Delta = 0\} \subset \mathbb{P}^4$ is the *Segre cubic*. The 15 hyperplanes in \mathbb{P}^4 defined by the 15 equations above (q_{+C}, \dots, p_{-3}) shall be referred as the *singular hyperplanes*.

The Segre cubic has 10 nodes, namely:

$$\begin{aligned}
& [1, 0, -2, -2, 2], [1, 0, -2, 2, -2], [1, 0, 2, -2, -2], [1, 0, 2, 2, 2], \\
& [0, -2, 1, 0, 0], [0, 2, 1, 0, 0], [0, -2, 0, 1, 0], [0, 2, 0, 1, 0], [0, -2, 0, 0, 1], \text{ and } [0, 2, 0, 0, 1].
\end{aligned}$$

We shall denote these 10 points by q_i , $i \in [1, \dots, 10]$, as ordered above. These 10 points have associated quartic in \mathbb{P}^3 , which turns out to be quadrics of multiplicity 2. We denote the quadric associated to the point q_i by Q_i . Explicitly they are:

$$\begin{aligned}
& x^2 - y^2 - z^2 + w^2 = 0, \quad x^2 - y^2 + z^2 - w^2 = 0, \quad x^2 + y^2 - z^2 - w^2 = 0, \quad x^2 + y^2 + z^2 + w^2 = 0, \\
& xy - zw = 0, \quad xy + zw = 0, \quad xz - yw = 0, \quad xz + yw = 0, \quad xw - yz = 0, \quad \text{and } xw + yz = 0.
\end{aligned}$$

It is known that for a general point on S_3 the corresponding surface is a Kummer surface ([Ekl10, Prop 2.2]). The following proposition links such Kummer surfaces with their singular points.

Proposition 2.3. Let $p = [x, y, z, w]$ be a point in $\mathbb{P}^3 \setminus \mathcal{L}$ and let

- $A = (yz + xw)(yz - xw)(xz + yw)(xz - yw)(zw + xy)(zw - xy),$
- $B = 2xyzw(-x^2 - y^2 + z^2 + w^2)(-x^2 + y^2 + z^2 - w^2)(x^2 - y^2 + z^2 - w^2)(x^2 + y^2 + z^2 + w^2),$
- $C = (yz + xw)(yz - xw)(xz + yw)(xz - yw)(x^4 + y^4 - z^4 - w^4),$

- $D = (yz + xw)(yz - xw)(zw + xy)(zw - xy)(-x^4 + y^4 - z^4 + w^4),$
- $E = (xz + yw)(xz - yw)(zw + xy)(zw - xy)(x^4 - y^4 - z^4 + w^4).$

Then the point $[A, B, C, D, E]$ lies on the Segre cubic and the associated Kummer surface has the 16 singular points $\{\gamma([x, y, z, w]) : \gamma \in \Gamma\}.$

Proof. Let $F = A(X^4 + Y^4 + Z^4 + W^4) + \dots + E(X^2W^2 + Z^2Y^2).$ By algebraic manipulation, the system of linear equations

$$\frac{\partial F}{\partial X}(p) = \frac{\partial F}{\partial Y}(p) = \frac{\partial F}{\partial Z}(p) = \frac{\partial F}{\partial W}(p) = 0$$

has a unique solution $[A, B, C, D, E] \in \mathbb{P}^4$ as given above. Note that in particular, such a point p defines uniquely the Kummer surface of which it is a singular point. \square

Proposition 2.4. *Let $[A, B, C, D, E]$ be a point on the Segre cubic not lying on one of the 15 singular hyperplanes. Then the associated surface's 16 singular points are $[x, y, z, w]$ where x, y, z and w solve the following equations*

- $az^8 + bz^6w^2 + cz^4w^4 + bz^2w^6 + aw^8 = 0,$ where $a = -A^2B^2,$ $b = 4(2AD - CE)(2AE - CD)$ and $c = 2(A^2B^2 - 2(E^2 + D^2)(4A^2 + C^2) + 16ACDE),$
- $(4A^2 - C^2)(Ez^2 - Dw^2)y^2 + A((4A^2 - C^2)(z^4 - w^4) + (E^2 - D^2)(z^4 + w^4)) + C(E^2 - D^2)z^2w^2 = 0,$
- $2(C^2 - 4A^2)xyzw + BCz^2w^2 + ABw^4 + ABz^4 = 0.$

Proof. Without loss of generality, we assume $w = 1.$ Then the first equation can be considered as a symmetric quartic polynomial with the variable $z^2,$ and hence z can be written as a radical function of $A, B, C, D, E,$ i.e.,

$$z = \pm \sqrt{\frac{u_{\pm} \pm \sqrt{u_{\pm}^2 - 4}}{2}}, \text{ where } u_{\pm} = \frac{-b \pm \sqrt{b^2 - 4a(c - 2a)}}{2a}.$$

Similarly, we can write x and y as radical functions of $A, B, C, D, E.$ Substituting the point $[x, y, z, 1]$ (written in terms of A, B, C, D, E) into the equations of Proposition 2.3, we get an equality. Since a point defines the Kummer surface uniquely, we must have that the point $[x, y, z, 1]$ is a singular point to $[A, B, C, D, E].$ \square

We explain why we need the hypothesis in the two previous propositions, namely taking a point in \mathbb{P}^3 away from \mathcal{L} and taking a point in \mathbb{P}^4 away from the singular hyperplanes. First we note that the intersection of one of the singular hyperplanes with the Segre cubic breaks down into 3 planes. For example

$$\begin{aligned} \{q_{+C} = 0\} \cap \{\Delta = 0\} &= \{q_{+C} = 0, q_{-C} = 0, A = 0\} \\ &\cup \{q_{+C} = 0, p_{-0} = 0, p_{+1} = 0\} \\ &\cup \{q_{+C} = 0, p_{+0} = 0, p_{-1} = 0\}. \end{aligned}$$

One can check that we get 15 planes this way, which we shall refer to as the 15 *Segre planes*.

Suppose the surface X_p is represented by a point p lying on one of the 15 Segre planes, that is it doesn't satisfy the hypothesis of Proposition 2.4. Then, one can calculate, that X_p does not have only 16 singular points, but rather two skew singular lines. Namely one of the 15 pairs of lines in $\mathcal{L}.$ On the other hand, consider the surface $X_p,$ with $p \in \mathbb{P}^4,$ which has the singular point $q \in \mathcal{L}.$ By Proposition 2.1 we know that either p lies on the Segre cubic or on one of the 15 singular hyperplanes. If p lies on the Segre cubic, then in fact p lies on one of the Segre planes. If p lies on a singular hyperplane and not on the Segre cubic, then q lies on 3 lines contained in $\mathcal{L}.$

Hence we have a one to one correspondence between the 15 pairs of skew lines of \mathcal{L} and the 15 Segre planes. Table 1 shows which Segre plane corresponds to which pair of lines.

Definition 2.5. Let Y be a quartic surface in $\mathbb{P}^3.$ We say that a plane T in \mathbb{P}^3 is a *trope* of Y if $Y \cap T$ is an irreducible conic counted with multiplicity two.

Lemma 2.6. *A quartic surface $Y \subset \mathbb{P}^3$ which has a trope T is necessarily singular.*

We now turn to the theorem from Eklund, [Ekl10, Thm 4.3], which started the idea of this paper.

Theorem 2.7. *A general K3 surface X from the family \mathcal{X} contains at least 320 smooth conics.*

Proof. The proof is adapted from [Ekl10, Thm 4.3], which we reproduce here as we will use some elements of the proof in the rest of this paper. For this proof if Y is a hypersurface, fix \tilde{Y} to be an equation defining Y .

Pick $p \in \mathbb{P}^4$ general and let q_i be a node of the Segre cubic S_3 (in particular fix i). We have that the associated surface to q_i is Q_i^2 , a quadric of multiplicity two. The line through p and q_i intersects S_3 in exactly one more point, call it p_i . Hence we have $\tilde{X}_p = \alpha (\tilde{Q}_i)^2 + \alpha' \tilde{X}_{p_i}$ for some $\alpha, \alpha' \in \overline{\mathbb{Q}}$. As p is general by Proposition 2.4, we have that the associated surface X_{p_i} is Kummer. Pick a singular point on X_{p_i} and consider its dual T . As T is a trope of X_{p_i} (see [Ekl10, pg 12] for more details) we have that $\tilde{X}_{p_i} = \mu(Q')^2 + \lambda \tilde{T}$, for some $\mu \in \overline{\mathbb{Q}}$, a cubic equation λ and a quadratic equation Q' . Hence, as equations,

$$\begin{aligned} \tilde{X}_p &= \alpha (\tilde{Q}_i)^2 + \alpha' \mu (Q')^2 + \alpha' \lambda \tilde{T} \\ &= (\sqrt{\alpha} \tilde{Q}_i + \sqrt{-\alpha' \mu} Q')(\sqrt{\alpha} \tilde{Q}_i - \sqrt{-\alpha' \mu} Q') + \alpha' \lambda \tilde{T}. \end{aligned} \quad (2)$$

So $X_p \cap T$ is the union of two conics. As the general member of the family does not contain any lines (see [Ekl10, Prop 2.3]), nor does it have a trope (Lemma 2.6), we have that the two conics are smooth and distinct.

Since a general Kummer surface of \mathcal{X} is determined by any of its tropes (i.e., by its singular points Proposition 2.3), all the tropes defined by using the 10 nodes q of S_3 are different. As two different planes cannot have a smooth conic in common, we conclude that we have constructed $10 \cdot 16 \cdot 2 = 320$ smooth conics on X_p . \square

3 The Galois Group

In this section we are going to shift away from working over $\overline{\mathbb{Q}}$ to working over number fields. Let K be a number field, and let p be a general point in \mathbb{P}_K^4 . Then the associated K3 surface, X_p , has 320 conics on it, so let L be the smallest number field over which those conics are defined. Note that it must be an extension of K . We want to work out the Galois group of the field of definition of the 320 conics. That is, we are interested in $\text{Gal}(L/K)$. To do this we will first find L . For this, by Theorem 2.7, it is sufficient to work out, for each conic, over what field α, α', μ and Q' are defined over (since Q_i is defined over \mathbb{Q} for all i).

Let $p = [A, B, C, D, E] \in \mathbb{P}^4$, we have that α and α' depend only on Q_i (or more specifically on the point q_i), while μ and Q' depend on the Q_i and the trope T (of which there are 16 choices once Q_i is fixed). Let α_i and α'_i be associated to Q_i and we first look at α_i, α'_i . Using the equations defining the line through the point p and the point q_i , and the cubic equation defining S_3 we can find the point p_i . Hence we write $[A_i, B_i, C_i, D_i, E_i] = X_{p,i}$ in terms of A, B, C, D and E . Since we know X_{q_i} , we can use simple algebra to calculate α_i and α'_i . We find that

$$\alpha_i = \Delta \beta_i^{-1} \text{ and } \alpha'_i = \begin{cases} \beta_i^{-1} & i \in [1, \dots, 4] \\ (4\beta_i)^{-1} & i \in [5, \dots, 10] \end{cases} \text{ where}$$

$$\begin{aligned} \beta_1 &= 12A^2 + \frac{1}{4}B^2 + 4A(C + D - E) - (C^2 + D^2 + E^2) + 2(CD - CE - DE) \\ \beta_2 &= 12A^2 + \frac{1}{4}B^2 + 4A(C - D + E) - (C^2 + D^2 + E^2) + 2(-CD + CE - DE) \\ \beta_3 &= 12A^2 + \frac{1}{4}B^2 + 4A(-C + D + E) - (C^2 + D^2 + E^2) + 2(-CD - CE + DE) \\ \beta_4 &= 12A^2 + \frac{1}{4}B^2 - 4A(C + D + E) + (C^2 + D^2 + E^2) + 2(CD + CE + DE) \end{aligned}$$

$$\begin{aligned}
\beta_5 &= -(AB + 2AC - DE) & \beta_6 &= AB - 2AC + DE \\
\beta_7 &= -(AB + 2AD - CE) & \beta_8 &= AB - 2AD + CE \\
\beta_9 &= -(AB + 2AE - CD) & \beta_{10} &= AB - 2AE + CD
\end{aligned}$$

and Δ is defined by Equation (1). In particular, letting \tilde{Y} be the equation defining the hypersurface Y as in Theorem 2.7, $\tilde{X}_p = \beta_i^{-1}(\Delta(\tilde{Q}_i)^2 + \tilde{X}_{p_i})$, with the $\frac{1}{4}$ factor absorbed in the equation defining X_{p_i} when needed. Hence, using Equation (2) and by rescaling with β , we have, for a fixed Q_i and T ,

$$X_p \cap T = \left\{ \left(\tilde{Q}_i + \sqrt{-\frac{\mu_i}{\Delta}} Q' \right) \left(\tilde{Q}_i - \sqrt{-\frac{\mu_i}{\Delta}} Q' \right) = 0, \tilde{X}_p = 0 \right\}$$

In particular, the field of definition of the pair of conics defined by T only depend on $\sqrt{-\frac{\mu_i}{\Delta}}$ and Q' . So let us fix an i , and set $X_{p_i} = [A_i, B_i, C_i, D_i, E_i]$ and let us fix T by choosing the singular point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$ on X_{p_i} . That is, T is defined by $r_{3,i}x + r_{2,i}y + r_{1,i}z + w = 0$. If we use the equations in Proposition 2.3 to rewrite the equation defining X_{p_i} in terms of $r_{3,i}, r_{2,i}$ and $r_{1,i}$, then substituting $w = -(r_{3,i}x + r_{2,i}y + r_{1,i}z)$ into X_{p_i} we find that $(Q')^2 = (a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_4yz)^2$ where

- $a_0 = (r_2r_3 - r_1) \cdot (r_2r_3 + r_1) \cdot (r_1r_3 - r_2) \cdot (r_1r_3 + r_2)$,
- $a_1 = (r_2r_3 - r_1) \cdot (r_2r_3 + r_1) \cdot (r_1r_2 - r_3) \cdot (r_1r_2 + r_3)$,
- $a_2 = (r_1r_3 - r_2) \cdot (r_1r_3 + r_2) \cdot (r_1r_2 - r_3) \cdot (r_1r_2 + r_3)$,
- $a_3 = r_3 \cdot r_2 \cdot (2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$,
- $a_4 = r_3 \cdot r_1 \cdot (2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$,
- $a_5 = r_2 \cdot r_1 \cdot (2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$.

Now as each trope T , and hence each associated Q' , of X_{p_i} are defined by Γ acting on the point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$ we have that the 16 conics Q' associated to X_{p_i} are defined over field $K(r_{1,i}, r_{2,i})$ (recall that $r_{3,i}$ is a K -linear combination of $r_{1,i}, r_{2,i}$, see Proposition 2.4).

Next to work out μ_i (depending on the singular point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$), we use the fact that (as equations) $X_{p_i} = \mu(Q')^2 + \lambda T$ and that Q' has no w terms, to find that

$$\mu_i = (A_i r_1^4 + C_i r_1^2 + A_i) \cdot a_2^{-2}.$$

Again we see that the action of Γ on $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$ will give us the other 15 μ 's. In particular, as the 15 other singular points have z -coordinates $\pm r_{1,i}, \pm \frac{1}{r_{1,i}}, \pm \frac{r_{2,i}}{r_{3,i}}, \pm \frac{r_{3,i}}{r_{2,i}}$, there are 3 other μ , namely $\frac{1}{r_{1,i}^2}\mu, \bar{\mu} = (A_i \frac{r_{2,i}^4}{r_{3,i}^4} + C_i \frac{r_{2,i}^2}{r_{3,i}^2} + A_i) \cdot \bar{a}_2^{-2}$ and $\frac{r_{3,i}^2}{r_{2,i}^2}\bar{\mu}$ (where \bar{a}_2 can be calculated, but will not be needed). Putting all of this together we have proven the following.

Proposition 3.1. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ and fix $i \in [1, \dots, 10]$. Let $[A_i, B_i, C_i, D_i, E_i] = p_i \in \mathbb{P}^4$ be the third point of intersection between the Segre cubic S_3 , and the line joining q_i and p . Then the 32 conics lying on X_p and associated to the point q_i (as per the construction in Theorem 2.7) are defined over*

$$K_i = K(r_{1,i}, r_{2,i}, r_{\mu,i}, \overline{r_{\mu,i}}) \quad (3)$$

where

$$r_{1,i} \text{ is a root of } ax^8 + bx^6 + cx^4 + bx^2 + a \quad (4)$$

$$r_{2,i} \text{ is a root of } d(Eir_{1,i}^2 - Di)x^2 + A_i(d(r_{1,i}^4 - 1) + e(r_{1,i}^4 + 1)) + C_i er_{1,i}^2 \quad (5)$$

$$r_{\mu,i} \text{ is a root of } x^2 + \frac{1}{\Delta}(A_i r_{1,i}^4 + C_i r_{1,i}^2 + A_i) \quad (6)$$

$$\overline{r_{\mu,i}} \text{ is a root of } x^2 + \frac{1}{\Delta}(A_i \overline{r_{1,i}^4} + C_i \overline{r_{1,i}^2} + A_i) \quad (7)$$

with $\bar{r}_{1,i} = \frac{r_{2,i}}{r_{3,i}}$ (which can be expressed in terms of $r_{1,i}$) and

$$\begin{aligned} a &= -A_i^2 B_i^2 \\ b &= 4(2A_i D_i - C_i E_i)(2A_i E_i - C_i D_i) \\ c &= 2(A_i^2 B_i^2 - 2(E_i^2 + D_i^2)(4A_i^2 + C_i^2) + 16A_i C_i D_i E_i) \\ d &= 4A_i^2 - C_i^2 \quad e = E_i^2 - D_i^2 \end{aligned}$$

Proposition 3.2. *Let X_p be a K3 surface in the family \mathcal{X} , for each $i \in [1, \dots, 10]$ define K_i as in Proposition 3.1. Then $\text{Gal}(K_i/K) \cong C_2^n$ for some $0 \leq n \leq 5$ (that is n copies of $\mathbb{Z}/2\mathbb{Z}$). In particular, $K_i = K(r_{1,i}, r_{2,i}, r_{\mu,i})$ (i.e., adjoining $\bar{r}_{\mu,i}$ is redundant).*

Proof. We are going to show that if the polynomials (4) to (7) are irreducible then $\text{Gal}(K_i/K) \cong C_2^5$. If any of the polynomials are not irreducible, then $\text{Gal}(K_i/K)$ is a subgroup of C_2^5 , and hence must be C_2^n for some $0 \leq n \leq 5$.

To do so we will use the resolvent method. Consider the group

$$\langle (12)(34)(56)(78), (13)(24)(57)(68), (15)(37)(26)(48) \rangle \leq S_8.$$

Note that this is the group of translations of $(\mathbb{Z}/2\mathbb{Z})^3$ (label the eight vertices of a fundamental cube 1 to 8), hence it is C_2^3 . Let x_1, \dots, x_8 be indeterminate variables, then S_8 acts on them by $x_i \mapsto x_{\sigma(i)}$. Note that the monomial $x_1 x_3 + x_2 x_4 + x_5 x_7 + x_6 x_8$ is C_2^3 -invariant, so we can construct the resolvent polynomial $R_{C_2^3} = \prod_{j=1}^8 (X - P_j)$ where P_j are the elements in the S_8 -orbit of $x_1 x_3 + x_2 x_4 + x_5 x_7 + x_6 x_8$.

We first consider the Galois group of $K(r_{1,i})$ over K , call it G . As polynomial (4) has as roots the eight different z coordinates of the 16 singular points, we have that the minimal polynomial of $r_{1,i}$ factors as

$$(x - r_{1,i})(x + r_{1,i}) \left(x - \frac{1}{r_{1,i}}\right) \left(x + \frac{1}{r_{1,i}}\right) (x - \bar{r}_{1,i})(x + \bar{r}_{1,i}) \left(x - \frac{1}{\bar{r}_{1,i}}\right) \left(x + \frac{1}{\bar{r}_{1,i}}\right).$$

If we substitute the x_j with the j th root of the minimal polynomial of $r_{1,i}$ (as ordered above), we find that

$$x_1 x_3 + x_2 x_4 + x_5 x_7 + x_6 x_8 = 4.$$

Hence in this case $R_{C_2^3}$ has a K -rational non-repeated root, so $G \subseteq C_2^3$. But since the minimal polynomial of $r_{1,i}$ is already of degree 8, we must have $G \cong C_2^3$. In fact G is generated by $r_{1,i} \mapsto -r_{1,i}$, $r_{1,i} \mapsto \frac{1}{r_{1,i}}$ and $r_{1,i} \mapsto \bar{r}_{1,i}$, denote them by σ_2, σ_3 and σ_4 respectively.

Next, we consider the Galois group of $K(r_{1,i}, r_{2,i})$ over K . We have that the minimal polynomial of $r_{2,i}$ is of degree 8 (either through direct calculation, or the fact that $r_{2,i}$ solves a quadratic in $r_{1,i}^2$ which itself solves a quartic). We can find all the conjugates of $r_{2,i}$, by noting that if we let $\sigma_2, \sigma_3, \sigma_4$ act on equation (5), we get with $\pm r_{2,i}$ a total of 8 conjugates. Furthermore, we know that $\pm r_{2,i}$ corresponds to the y -coordinate of the singular points which have z -coordinate $r_{1,i}$. Similarly, $\sigma_3(\pm r_{2,i})$ corresponds to the y -coordinate of the singular points which have z -coordinate $\frac{1}{r_{1,i}}$. Hence we know that the minimal polynomial of $r_{2,i}$ factors as

$$(x - r_{2,i})(x + r_{2,i}) \left(x - \frac{1}{r_{2,i}}\right) \left(x + \frac{1}{r_{2,i}}\right) (x - \bar{r}_{2,i})(x + \bar{r}_{2,i}) \left(x - \frac{1}{\bar{r}_{2,i}}\right) \left(x + \frac{1}{\bar{r}_{2,i}}\right),$$

where $\bar{r}_{2,i} = \frac{r_{1,i}}{r_{1,i}^2} r_{2,i}$. As above, we can see that the Galois group of $K(r_{2,i})$ over K is C_2^3 , and in particular, we now know that the $K(r_{1,i}, r_{2,i})/K$ is Galois. After having made some choice of sign on $\sigma_i(r_{2,i})$ for $2 \leq i \leq 4$, it is not hard to see that in fact $\text{Gal}(K(r_{1,i}, r_{2,i})/K) \cong C_2^4$ generated by $\sigma_1, \sigma_2, \sigma_3$ and σ_4 , where $\sigma_1(r_{1,i}) = r_{1,i}$ and $\sigma_1(r_{2,i}) = -r_{2,i}$.

Finally we look at $K(r_{\mu,i}, \bar{r}_{\mu,i})$, and first note that $r_{\mu,i}$ and $\bar{r}_{\mu,i}$ have the same minimal polynomial over K . In fact, we have that the minimal polynomial of $r_{\mu,i}$ factors as

$$(x - r_{\mu,i})(x + r_{\mu,i}) \left(x - \frac{r_{\mu,i}}{r_{1,i}^2}\right) \left(x + \frac{r_{\mu,i}}{r_{1,i}^2}\right) (x - \bar{r}_{\mu,i})(x + \bar{r}_{\mu,i}) \left(x - \frac{\bar{r}_{\mu,i}}{\bar{r}_{1,i}^2}\right) \left(x + \frac{\bar{r}_{\mu,i}}{\bar{r}_{1,i}^2}\right).$$

In this case, if we substitute the x_j with the j th root of the minimal polynomial of $r_{\mu,i}$ (as ordered above), we find that

$$\begin{aligned} x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8 &= 2 \left(\frac{r_{\mu,i}^2}{r_{1,i}^2} + \frac{\bar{r}_{\mu,i}^2}{\bar{r}_{1,i}^2} \right) \\ &= -\frac{2}{\Delta} \left(2C_1 + A_1 \left(r_{1,i}^2 + \frac{1}{r_{1,i}^2} + \bar{r}_{1,i}^2 + \frac{1}{\bar{r}_{1,i}^2} \right) \right). \end{aligned}$$

Since $r_{1,i}^2$ solves a quartic polynomial whose other roots are $\frac{1}{r_{1,i}^2}, \bar{r}_{1,i}^2, \frac{1}{\bar{r}_{1,i}^2}$, we have that the above expression is in K . So $R_{H'}$ has a K -rational non-repeated root, hence $H \subseteq H' \cong C_2^3$. But since the minimal polynomial of $r_{\mu,i}$ is already of degree 8, we must have $H \cong C_2^3$ and $K(r_{\mu,i}, \bar{r}_{\mu,i}) \cong K(r_{\mu,i})$.

Hence since $[K_i : K] = 2 \cdot 2 \cdot 8 = 32$, so we are looking for a group of order 32, which has C_2^4 as a subgroup. Note that the element fixing $r_{1,i}, r_{2,i}$ and sending $r_{\mu,i} \mapsto \frac{1}{r_{\mu,i}}$, call it σ_5 , has order 2, but is not in the subgroup $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle \cong C_2^4$ (again, after having made some choice of sign on $\sigma_i(r_{\mu,i})$). Furthermore, one can check that σ_5 commutes with σ_i for $1 \leq i \leq 4$ (after having extended σ_i properly on K_i). Hence we have that $\text{Gal}(K_i/K) \cong C_2^5$. \square

The following lemma will allow us to find another way of expressing K_i , which will help us finding L . While this lemma is quite standard, the proof has been included as it details how one can construct the field isomorphic of K_i .

Lemma 3.3. *If $\text{Gal}(L/K) \cong C_2^n$ for some n , then there exists $\Delta_1, \dots, \Delta_n \in K$ whose image are linearly independent in the \mathbb{F}_2 -vector space $K^*/(K^*)^2$, such that $L \cong K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_n})$.*

Proof. Let $\text{Gal}(L/K) = \langle \sigma_1, \dots, \sigma_n | \sigma_i^2 = (\sigma_i \sigma_j)^2 = 1 \rangle \cong C_2^n$ and let

$$\tilde{\sigma}_i = \langle \sigma_1, \dots, \sigma_{i-1}, \sigma_{i+1}, \dots, \sigma_n \rangle \cong C_2^{n-1}$$

(for each $i \in \{1, \dots, n\}$). Since, for each i , we have that $[\text{Gal}(L/K) : \tilde{\sigma}_i] = 2$, the fixed field of $\tilde{\sigma}_i$, $L^{\tilde{\sigma}_i}$, is a degree 2 extension of K . Hence $L^{\tilde{\sigma}_i} = K(\sqrt{\Delta_i})$ for some $\Delta_i \in K$.

We prove that $[K(\sqrt{\Delta_i})(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_{i-1}}) : K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_{i-1}})] = 2$ by showing that $\sqrt{\Delta_i} \notin K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_{i-1}})$. Suppose, that this was the case, then by considering minimal polynomial, we can show that $\sqrt{\Delta_i} = \alpha \sqrt{\Delta_{i_1} \dots \Delta_{i_s}}$ for some $\alpha \in K$ and subset $\{i_1, \dots, i_s\}$ of $\{1, \dots, i-1\}$, i.e., Δ_i is not linearly independent of $\Delta_1, \dots, \Delta_{i-1}$ in $K^*/(K^*)^2$. Hence $K(\sqrt{\Delta_i}) \cong K(\sqrt{\Delta_{i_1} \dots \Delta_{i_s}})$ and $\sigma_{i_1} \in \tilde{\sigma}_i$ fixes $\sqrt{\Delta_{i_1} \dots \Delta_{i_s}}$. But since $\sigma_{i_1} \in \tilde{\sigma}_j$ for $j \in \{i_2, \dots, i_s\}$, we also have that σ_{i_1} fixes $\sqrt{\Delta_j}$. So $\sqrt{\Delta_{i_1} \dots \Delta_{i_s}} = \sigma_{i_1}(\sqrt{\Delta_{i_1} \dots \Delta_{i_s}}) = \sigma_{i_1}(\sqrt{\Delta_{i_1}}) \sqrt{\Delta_{i_2} \dots \Delta_{i_s}}$, hence σ_{i_1} fixes $\sqrt{\Delta_{i_1}}$. This is a contradiction, since then $K(\sqrt{\Delta_{i_1}})$ is the fixed field of $\tilde{\sigma}_{i_1} \times \langle \sigma_{i_1} \rangle = \text{Gal}(L/K)$.

As $L^{\tilde{\sigma}_i} \subset L$, we have that $\sqrt{\Delta_i} \in L$. So $K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_n}) \subset L$, but by the previous paragraph and the tower law, we also have $[K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_n}) : K] = 2^n$. Hence $L \cong K(\sqrt{\Delta_1}, \dots, \sqrt{\Delta_n})$. \square

This means, that for each of the fields K_i , we can find an isomorphic field of the form $K(\sqrt{\Delta_{1,i}}, \dots, \sqrt{\Delta_{5,i}})$, and the compositum, i.e. L , will be $K(\sqrt{\Delta_{1,1}}, \dots, \sqrt{\Delta_{5,10}})$.

Proposition 3.4. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ not on the Segre cubic or singular hyperplanes, then the 32 conics lying on X_p defined by the point q_1 are defined over the field*

$$K_1 \cong K \left(\sqrt{\Delta_{q+CP-0P+1}}, \sqrt{\Delta_{q+CP+0P-1}}, \sqrt{\Delta_{q+DP+0P-2}}, \sqrt{\Delta_{q+DP-0P+2}}, \sqrt{-\Delta_{q-EP+1P-2}} \right).$$

Proof. We can use Lemma 3.3 to construct K_1 . We have $\text{Gal}(K_1/K) = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \rangle$ where σ_j acts on $r_{1,1}, r_{2,1}, r_{\mu,1}$ according to the following table

	σ_1	σ_2	σ_3	σ_4	σ_5
$r_{1,1}$	$-r_{1,1}$	$\frac{1}{r_{1,1}}$	$\bar{r}_{1,1}$	$r_{1,1}$	$r_{1,1}$
$r_{2,1}$	$r_{2,1}$	$\bar{r}_{2,1}$	$\frac{1}{r_{2,1}}$	$-r_{2,1}$	$r_{2,1}$
$r_{\mu,1}$	$r_{\mu,1}$	$\frac{r_{\mu,1}}{r_{1,1}^2}$	$\bar{r}_{\mu,1}$	$r_{\mu,1}$	$-r_{\mu,1}$

(where $\bar{r}_{2,1} = \frac{r_{1,1}}{\bar{r}_{1,1}} r_{2,1}$). We calculate the fixed field of $\tilde{\sigma}_1 = \langle \sigma_2, \dots, \sigma_5 \rangle$ by considering the expression $r_{1,1} + \frac{1}{r_{1,1}} + \bar{r}_{1,1} + \frac{1}{\bar{r}_{1,1}}$ which is fixed under σ_j , $j \in \{2, \dots, 5\}$ but not under σ_1 . Hence, upon calculating the discriminant of the minimal polynomial (after checking it is quadratic) of such an expression, we have that the fixed field of $\tilde{\sigma}_1$ is $K\left(\sqrt{p_{-0}p_{+1}p_{+2}p_{-2}q_{+D}(-q-E)}\right)$. Similarly we can use the following expressions to calculate the respective fixed fields:

- $r_{1,1}^2 + \bar{r}_{1,1}^2$ for $\tilde{\sigma}_2$ giving $K\left(\sqrt{p_{+1}p_{-1}p_{+2}p_{-2}}\right)$,
- $r_{1,1}^2 + \frac{1}{r_{1,1}^2}$ for $\tilde{\sigma}_3$ giving $K\left(\sqrt{p_{+0}p_{-0}p_{+1}p_{-1}}\right)$,
- $r_{2,1} + \frac{1}{r_{2,1}} + \bar{r}_{2,1} + \frac{1}{\bar{r}_{2,1}}$ for $\tilde{\sigma}_4$ giving $K\left(\sqrt{p_{+0}p_{+1}p_{-1}p_{-2}q_{+C}(-q-E)}\right)$,
- $r_{\mu,1} + \frac{r_{\mu,1}}{r_{1,1}^2} + \bar{r}_{\mu,1} + \frac{\bar{r}_{\mu,1}}{\bar{r}_{1,1}^2}$ for $\tilde{\sigma}_5$ giving $K\left(\sqrt{\Delta p_{+0}p_{-0}q_{+C}q_{+D}(-q-E)}\right)$.

Putting all of this together and rearranging, we get the required result. \square

Theorem 3.5. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ be a general point not lying on the Segre cubic or the 15 singular hyperplanes and let L be the field where the 320 conics of X_p are defined. Then $\text{Gal}(L/K) \cong C_2^{10}$.*

Proof. The first step is to calculate K_i for $i \in \{2, \dots, 10\}$ in terms of squares roots of elements in K . This is done by doing same calculations as the above proposition with different q_i (and hence $r_{1,i}$, $r_{2,i}$, $r_{\mu,i}$). We find, up to rearrangements

$$\begin{aligned}
K_2 &= K\left(\sqrt{\Delta q_{+C}p_{-0}p_{+1}}, \sqrt{\Delta q_{+C}p_{+0}p_{-1}}, \sqrt{\Delta q_{+E}p_{+0}p_{-3}}, \sqrt{\Delta q_{+E}p_{-0}p_{+3}}, \sqrt{-\Delta q_{-D}p_{+1}p_{-3}}\right), \\
K_3 &= K\left(\sqrt{\Delta q_{+D}p_{-0}p_{+2}}, \sqrt{\Delta q_{+D}p_{+0}p_{-2}}, \sqrt{\Delta q_{+E}p_{+0}p_{-3}}, \sqrt{\Delta q_{+E}p_{-0}p_{+3}}, \sqrt{-\Delta q_{-C}p_{+2}p_{-3}}\right), \\
K_4 &= K\left(\sqrt{-\Delta q_{-D}p_{+1}p_{-3}}, \sqrt{-\Delta q_{-D}p_{-1}p_{+3}}, \sqrt{-\Delta q_{-E}p_{-1}p_{+2}}, \sqrt{-\Delta q_{-E}p_{+1}p_{-2}}, \sqrt{-\Delta q_{-C}p_{+2}p_{-3}}\right), \\
K_5 &= K\left(\sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{-\Delta A q_{+D}q_{-D}}, \sqrt{\Delta q_{+D}p_{+0}p_{-2}}, \sqrt{\Delta q_{+E}p_{+0}p_{-3}}, \sqrt{-\Delta q_{-E}p_{+1}p_{-2}}\right), \\
K_6 &= K\left(\sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{-\Delta A q_{+D}q_{-D}}, \sqrt{\Delta q_{+D}p_{-0}p_{+2}}, \sqrt{\Delta q_{+E}p_{-0}p_{+3}}, \sqrt{-\Delta q_{-E}p_{-1}p_{+2}}\right), \\
K_7 &= K\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{\Delta q_{+C}p_{+0}p_{-1}}, \sqrt{\Delta q_{+E}p_{+0}p_{-3}}, \sqrt{-\Delta q_{-E}p_{+1}p_{-2}}\right), \\
K_8 &= K\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{\Delta q_{+C}p_{-0}p_{+1}}, \sqrt{\Delta q_{+E}p_{-0}p_{+3}}, \sqrt{-\Delta q_{-E}p_{-1}p_{+2}}\right), \\
K_9 &= K\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta q_{+D}q_{-D}}, \sqrt{\Delta q_{+C}p_{+0}p_{-1}}, \sqrt{\Delta q_{+D}p_{+0}p_{-2}}, \sqrt{-\Delta q_{-D}p_{+1}p_{-3}}\right), \\
K_{10} &= K\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta q_{+D}q_{-D}}, \sqrt{\Delta q_{+C}p_{-0}p_{+1}}, \sqrt{\Delta q_{+D}p_{-0}p_{+2}}, \sqrt{-\Delta q_{-D}p_{-1}p_{+3}}\right).
\end{aligned}$$

Then as the 320 conics of X_p are defined over the compositum of K_1, \dots, K_{10} , we see that L is the field extension

$$\begin{aligned}
&K\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta A q_{+D}q_{-D}}, \sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{\Delta q_{+C}p_{+0}p_{-1}}, \sqrt{\Delta q_{+C}p_{-0}p_{+1}}, \right. \\
&\quad \left. \sqrt{\Delta q_{+D}p_{+0}p_{-2}}, \sqrt{\Delta q_{+D}p_{-0}p_{+2}}, \sqrt{\Delta q_{+E}p_{+0}p_{-3}}, \sqrt{\Delta q_{+E}p_{-0}p_{+3}}, \sqrt{-\Delta q_{-C}p_{+2}p_{-3}}\right).
\end{aligned}$$

Now for a general point p , the 10 square root are distinct and not in K , i.e., $[L : K] = 2^{10}$ (this holds, for example, in the case $K = \mathbb{Q}$ and $[A, B, C, D, E] = [1, 87, 15, 39, 21]$). Hence $\text{Gal}(L/K) \cong C_2^{10}$. \square

4 Monodromy Group

Note that the field of definition of the 320 conics involved equations defining when a surface X in \mathcal{X} is singular (cf Theorem 3.5 and Proposition 2.1). We explain this by studying the Monodromy group of the conics over a general non-singular K3 surface in \mathcal{X} . First we briefly recall what the Monodromy group is.

Let Z be an algebraic variety with $\pi : Z \rightarrow X$ a surjective finite map of degree $d > 0$. Let $p \in X$ be a general point of X and $\pi^{-1}(p) = \{q_0, \dots, q_{d-1}\}$ be its fibre. Let $U \subset X$ be a suitable small Zariski open subset of X and $V = \pi^{-1}(U)$. Then for any loop $\lambda : [0, 1] \rightarrow U$ based at p , and any point $q_i \in \pi^{-1}(p)$, there exists a unique path $\tilde{\lambda}_i$ in V such that $\pi(\tilde{\lambda}_i) = \lambda$ and $\tilde{\lambda}_i(0) = q_i$. So we may define a permutation σ_λ of $\pi^{-1}(p)$ by sending each point q_i to $\tilde{\lambda}_i(1) = q_j$ (for some j). Since ϕ_λ only depends on the homotopy class of λ , we have a homomorphism $\pi_1(U, p) \rightarrow S_d$. The image of this homomorphism is called the *Monodromy group* of the map π .

In our case we want the variety Z to parametrise the quartic surfaces of \mathcal{X} with the 320 conics on them, and study the permutations of the conics as we draw loops on \mathcal{X} . We first study a simpler problem, namely we will look at the Monodromy group of the 16 planes defined by the point $q_1 = [1, 0, -2, -2, 2] \in \mathbb{P}^4$. We shall denote this set of 16 planes by T_1 , (they are the tropes described in the proof of Theorem 2.7)

Lemma 4.1. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ be a general point not lying on the Segre cubic or the 15 singular hyperplanes. The set T_1 is $\{\gamma(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w) | \gamma \in \Gamma\}$ where*

- $r_{0,1} = 2^3 B \sqrt{-q_{+D}q_{+C}q_{-E}}$,
- $r_{1,1} = \sqrt{q_{+C}} (\sqrt{p_{-2}p_{-0}p_{+2}p_{+1}} + p_{+2}\sqrt{p_{+1}p_{+0}} + \sqrt{p_{-2}p_{-1}p_{+2}p_{+0}} + p_{-2}\sqrt{p_{-1}p_{-0}})$,
- $r_{2,1} = \sqrt{q_{+D}} (\sqrt{p_{-1}p_{-0}p_{+1}p_{+2}} + p_{+1}\sqrt{p_{+2}p_{+0}} + \sqrt{p_{-1}p_{-2}p_{+1}p_{+0}} + p_{-1}\sqrt{p_{-2}p_{-0}})$,
- $r_{3,1} = -\sqrt{-q_{-E}} (\sqrt{p_{-0}p_{-1}p_{+0}p_{+2}} + p_{+0}\sqrt{p_{+2}p_{+1}} + \sqrt{p_{-0}p_{-2}p_{+0}p_{+1}} + p_{-0}\sqrt{p_{-2}p_{-1}})$,

where $p_{\pm i}, q_{\pm \alpha}$ are the equations in Proposition 2.1.

Proof. Theorem 3.1 already gives an expression for the planes, but we can use the isomorphism $K(r_{1,1}, r_{2,1}, r_{\mu,1}) \cong K(\sqrt{\Delta_{q+C}p_{-0}p_{+1}}, \sqrt{\Delta_{q+C}p_{+0}p_{-1}}, \sqrt{\Delta_{q+D}p_{+0}p_{-2}}, \sqrt{\Delta_{q+D}p_{-0}p_{+2}}, \sqrt{-\Delta_{q-E}p_{+1}p_{-2}})$ to rewrite the singular point $[r_{3,1}, r_{2,1}, r_{1,1}, 1]$ in terms of linear combinations of square roots.

That is, let r be any of those roots, we know that r solves a degree 8 polynomial, whose terms are all even and $K(r) \cong K(\sqrt{\Delta_1}, \sqrt{\Delta_2}, \sqrt{\Delta_3})$ for some Δ_i 's. So let $r = a_0 + a_1\sqrt{\Delta_1} + a_2\sqrt{\Delta_2} + \dots + a_7\sqrt{\Delta_1\Delta_2\Delta_3}$. The Galois group of $K(\sqrt{\Delta_1}, \sqrt{\Delta_2}, \sqrt{\Delta_3})$ is naturally generated by $\sigma_1, \sigma_2, \sigma_3$ where $\sigma_i(\sqrt{\Delta_j}) = \begin{cases} -\sqrt{\Delta_j} & i = j \\ \sqrt{\Delta_j} & j \neq i \end{cases}$. On one hand we know that the minimal polynomial of r factorises as

$$(x - r)(x + r) \left(x - \frac{1}{r}\right) \left(x + \frac{1}{r}\right) (x - \bar{r})(x + \bar{r}) \left(x - \frac{1}{\bar{r}}\right) \left(x + \frac{1}{\bar{r}}\right),$$

and on the other hand, it factorises

$$\prod_{g \in \text{Gal}(K(r)/K)} (x - g(a_0 + a_1\sqrt{\Delta_1} + \dots + a_7\sqrt{\Delta_1\Delta_2\Delta_3})).$$

Now we have $0 = r - r + \dots + \frac{1}{\bar{r}} - \frac{1}{\bar{r}} = 8a_0$, hence $a_0 = 0$. Without loss of generality, suppose that $\sigma_1(r) = -r$, then $0 = r - r = a_2\sqrt{\Delta_2} + a_3\sqrt{\Delta_3} + a_6\sqrt{\Delta_2\Delta_3}$, so $r = \sqrt{\Delta_1}(a_1 + a_4\sqrt{\Delta_2} + a_5\sqrt{\Delta_3} + a_7\sqrt{\Delta_2\Delta_3})$. If we suppose $\sigma_2(r) = \frac{1}{r}$ and $\sigma_3(r) = \bar{r}$, then note that

$$\begin{aligned} r + \frac{1}{r} + \bar{r} + \frac{1}{\bar{r}} &= 4a_1\sqrt{\Delta_1}, \\ r - \frac{1}{r} + \bar{r} - \frac{1}{\bar{r}} &= 4a_4\sqrt{\Delta_1\Delta_2}, \\ r + \frac{1}{r} - \bar{r} - \frac{1}{\bar{r}} &= 4a_5\sqrt{\Delta_1\Delta_3}, \\ r - \frac{1}{r} - \bar{r} + \frac{1}{\bar{r}} &= 4a_7\sqrt{\Delta_1\Delta_2\Delta_3}. \end{aligned}$$

Hence we can easily work out a_1, a_4, a_5, a_7 , and therefore r , by calculating the minimal polynomial of the above 4 expressions. Applying that theory to $r_{3,1}$, $r_{2,1}$ and $r_{1,1}$ in turn, after some rearrangement, we get the required result. \square

Hence, to study the Monodromy group of those 16 planes on a K3 surface, we need the object \mathcal{Z} defined by

$$\{([A, B, C, D, E], [a, b, c, d]) \mid [a, b, c, d] \in \{\gamma([r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}]) : \gamma \in \Gamma\}\} \subset \mathbb{P}^4 \times \mathbb{P}^3.$$

Note that we set up $\mathbb{P}_{[a,b,c,d]}^3$ to be the dual of $\mathbb{P}_{[x,y,z,w]}^3$, that is a point $[a, b, c, d] \in \mathbb{P}_{[a,b,c,d]}^3$ represent the plane $ax + by + cz + dw = 0$ in $\mathbb{P}_{[x,y,z,w]}^3$ which intersects $X_{[A,B,C,D,E]}$ in two conics. Now by the above lemma, $r_{i,1}$ involves square roots and hence \mathcal{Z} is not a variety. So instead of looking at the planes defined by the point $[A, B, C, D, E]$, we will look at the points $[A, B, C, D, E]$ that can be defined by a given plane. Pick a point $[a, b, c, d] \in \mathbb{P}^3$ and let

- $A_1 = (bc + ad)(bc - ad)(ac + bd)(ac - bd)(cd + ab)(cd - ab)$,
- $B_1 = 2abcd(-a^2 - b^2 + c^2 + d^2)(-a^2 + b^2 + c^2 - d^2)(a^2 - b^2 + c^2 - d^2)(a^2 + b^2 + c^2 + d^2)$,
- $C_1 = (bc + ad)(bc - ad)(ac + db)(ac - bd)(a^4 + b^4 - c^4 - d^4)$,
- $D_1 = (bc + ad)(bc - ad)(cd + ab)(cd - ab)(-a^4 + b^4 - c^4 + d^4)$,
- $E_1 = (ac + db)(ac - bd)(cd + ab)(cd - ab)(a^4 - b^4 - c^4 + d^4)$.

Then $[A, B, C, D, E]$ defines a surface which has the plane $ax + by + cz + dy = 0$ given by the point q_1 if and only if the point $[A, B, C, D, E]$ lies on the line joining $[A_1, B_1, C_1, D_1, E_1]$ and q_1 . So let us consider the variety \mathcal{V} defined by

$$\begin{aligned} \{E_1(2A + D) + D_1(2A - E) - 2A_1(E + D) = C_1(2A + D) - D_1(2A + C) + 2A_1(D - C) \\ = B_1(2A + D) - D_1B - 2A_1B = 0\} \subseteq \mathbb{P}_{[A,B,C,D,E]}^4 \times \mathbb{P}_{[a,b,c,d]}^3. \end{aligned}$$

The variety \mathcal{V} has the following properties:

- over any point of the dual of \mathcal{L} , we have a copy of $\mathbb{P}_{[A,B,C,D,E]}^4$,
- over the conic $a^2 - b^2 - c^2 + d^2$, we have a copy of $\mathbb{P}_{[A,B,C,D,E]}^4$,
- over the point $[1 : 0 : -2 : -2 : 2]$, we have a copy of $\mathbb{P}_{[a:b:c:d]}^3$,
- everywhere else, the variety \mathcal{V} coincides with to \mathcal{Z} .

We note that the dual of $\mathcal{L} \subset \mathbb{P}_{[x,y,z,w]}^3$ is itself $\mathcal{L} \subset \mathbb{P}_{[a,b,c,d]}^3$ (by making the correspondence $x \leftrightarrow a, \dots, w \leftrightarrow d$), this is because each L_i is dual to \bar{L}_i . Similarly, we have that each quadric $Q_i \subset \mathbb{P}_{[x,y,z,w]}^3$ can be identified with itself $Q_i \subset \mathbb{P}_{[a,b,c,d]}^3$.

Lemma 4.2. *Let $\mathcal{V} \subset \mathbb{P}_{[A,B,C,D,E]}^4 \times \mathbb{P}_{[a:b:c:d]}^3$ be as above and p_1, p_2 the projective maps $\mathcal{V} \mapsto \mathbb{P}_{[a,b,c,d]}^3$ and $\mathcal{V} \mapsto \mathbb{P}_{[A,B,C,D,E]}^4$ respectively. Then the projective map $p_2 : \mathcal{V} \rightarrow \mathbb{P}_{[A:B:C:D:E]}^4$ is smooth away from the points p such that $p_1(p)$ lies on 10 quadrics Q_i , or $p_2(p)$ is the point $q_1 = [1 : 0 : -2 : -2 : 2]$.*

Proof. Note that the union of the 15 pairs of lines \mathcal{L} are contained in the 10 quadrics Q_i . Away from these 15 pairs of lines, once we have fixed $[a : b : c : d]$, we have that the point $[A : B : C : D : E]$ is of the form

$$[\mu a_p + (1 - \mu), \mu b_p, \mu c_p - 2(1 - \mu), \mu d_p - 2(1 - \mu), \mu e_p + 2(1 - \mu)]$$

for some $\mu \in K$. So we want to show that the Jacobian matrix

$$\left(\frac{\partial g_i}{\partial x_j} \Big|_{A=\mu a_p+(1-\mu), B=\mu b_p, C=\mu c_p-2(1-\mu), D=\mu d_p-2(1-\mu), E=\mu e_p+2(1-\mu)} \right)_{i,j}$$

has rank 3. This is equivalent to showing that the determinant of at least one of the 4 matrices obtained from deleting a row in the Jacobian is non-zero. We can calculate that the 4 determinants are $\mu^3 aF$, $\mu^3 bF$, $\mu^3 cF$, $\mu^3 dF$ where

$$F = (bc - ad)^4 (bc + ad)^4 (ac - bd)^2 (ac + bd)^2 (ab - cd)^4 (ab + cd)^4 \cdot (a^2 - b^2 - c^2 + d^2)^6 (a^2 - b^2 + c^2 - d^2)^2 (a^2 + b^2 - c^2 - d^2)^4 (a^2 + b^2 + c^2 + d^2)^2.$$

Note that F is a product of the 10 quadratics, Q_i , and hence cannot be 0. While if $\mu = 0$, then the surface $[A, B, C, D, E]$ is $[1 : 0 : -2 : -2 : 2]$. If $F \neq 0$ and $\mu \neq 0$, then one of the 4 determinants must be non-zero, hence the projection map is smooth at that place. \square

So we only need to worry about points lying on one of the 10 quadrics Q_i . Since a point on it either lies on \mathcal{L} or the point gives rise to one of the 10 nodes, from our construction (and not the variety \mathcal{V}), such cases correspond to the K3 surface lying on one of the 15 singular hyperplanes. Hence to study the Monodromy group of 16 planes defined by q_1 , we need to look at the 15 singular hyperplanes. We will study this on the level of the object \mathcal{Z} and not the variety \mathcal{V} .

Proposition 4.3. *There exists a loop in \mathbb{P}^4 that:*

1. *goes around the singular hyperplane q_{+C} and avoids all of the 15 singular hyperplanes,*
2. *changes the sign of $\sqrt{-q_{+C}}$, which in turn sends the plane $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$ to the plane $\gamma_3(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w) = 0$. (Where $\gamma_3 \in \Gamma < \Omega$ as define in Section 2).*

Proof. Pick a point $[A : B : C : D : E] \in \mathbb{P}^4$ which does not represent a singular K3 surface, and note that $C \in K \subset \mathbb{C}$ can be written uniquely as $-2A + re^{i\phi}$ for some $r \in \mathbb{R}_{>0}$ and $\phi \in [0, 2\pi)$. Define a loop $\tilde{\lambda}(t) = [\tilde{\lambda}_A(t), \tilde{\lambda}_B(t), \tilde{\lambda}_C(t), \tilde{\lambda}_D(t), \tilde{\lambda}_E(t)]$ ($0 \leq t \leq 3$) as $\tilde{\lambda}_C = -2A + f(t)$ and $\tilde{\lambda}_j = j$ for all $t \in [0, 3]$ and $j \in \{A, B, D, E\}$, where f is composed of the following 3 segments:

$$f(t) = \begin{cases} (\rho t + r(1-t))e^{i\phi} & t \in [0, 1] \\ \rho e^{i\phi + i(t-1)2\pi} & t \in [1, 2] \\ (\rho(3-t) + r(t-2))e^{i\phi} & t \in [2, 3] \end{cases}$$

and where $\rho \in \mathbb{R}_{>0}$ satisfies

$$\rho < \min \{|B + 2D + 2E|, |-B + 2D + 2E|, |8A + B + 2D - 2E|, |8A - B + 2D - 2E|\}.$$

Now consider the point $[r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}]$ (where we have fixed a root for each square roots) under this loop. As t changes, the 10 equations occurring in $r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}$ are affected in the following ways:

- $q_{+C} = f(t)$,
- q_{+D}, q_{-E}, B all stay the same,
- $p_{+0} = -p_{+1} = B + 2D + 2E + 2f(t)$,
- $p_{-0} = -p_{+1} = -B + 2D + 2E + 2f(t)$,
- $p_{+2} = 8A + B + 2D - 2E - 2f(t)$,
- $p_{-2} = 8A - B + 2D - 2E - 2f(t)$.

For ease of argument we assume that for $0 \leq t \leq 1$, none of $\{p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2}, p_{-2}\}$ are 0 (if they are, the argument can be changed by slightly curving the first segment instead of using a straight line). So for the first segment, we can see that nothing remarkable happens. During the second segment, we have chosen ρ small enough so that none of $p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2}$ and p_{-2} are 0, but we see that $\sqrt{q_{+C}}$ is affected. Indeed, if we choose the square root of $e^{i\phi}$ to be $e^{\frac{i\phi}{2}}$, we see that $\sqrt{q_{+C}} = \sqrt{\rho} e^{\frac{i\phi}{2} + i(t-1)\pi}$. Hence at $t = 1$, $\sqrt{q_{+C}}$ is positive, but at $t = 2$ the sign has changed. Note that the third segment is the same as the first segment but backwards.

Finally, one can see that by changing the sign of $\sqrt{q_{+C}}$, we have $r_{0,1} \mapsto -r_{0,1}$, $r_{1,1} \mapsto -r_{1,1}$, $r_{2,1} \mapsto r_{2,1}$ and $r_{3,1} \mapsto r_{3,1}$. Hence the plane $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$ gets mapped to the plane $r_{0,1}x + r_{1,1}y - r_{2,1}z - r_{3,1}w = 0 = \gamma_3(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$. \square

A very similar argument works for the singular hyperplanes q_{+D} , q_{-E} , p_{+0} , p_{-0} , p_{+1} , p_{-1} , p_{+2} and p_{-2} . For the singular hyperplanes A , q_{-C} , q_{-D} , q_{+E} , p_{+3} and p_{-3} , we note that either $[r_0, r_1, r_2, r_3]$ are completely unaffected, or see by direct calculations that we still have 16 different planes when plucking in $A = 0$, or $q_{-C} = 0, \dots$ etc.

Notation. Out of the 15 singular hyperplanes, the point q_1 lies on 9 of them, namely q_{+C} , q_{+D} , q_{-E} , p_{+0} , p_{-0} , p_{+1} , p_{-1} , p_{+2} and p_{-2} . We shall denote this set by Σ_{q_1} .

Proposition 4.4. *The Monodromy group of set T_1 is isomorphic to Γ and hence C_2^4 .*

Proof. By the above discussion, the permutations of the 16 planes come from changing the sign of the square roots $\sqrt{\Delta_i}$ for $\Delta_i \in \Sigma_{q_1}$. By direct calculation, we see that:

- $\sqrt{q_{+C}} \mapsto -\sqrt{q_{+C}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_3(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{q_{+D}} \mapsto -\sqrt{q_{+D}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{-q_{-E}} \mapsto -\sqrt{-q_{-E}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_3\gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{+0}} \mapsto -\sqrt{p_{+0}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_1\gamma_2(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{-0}} \mapsto -\sqrt{p_{-0}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_1\gamma_2\gamma_3\gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{+1}} \mapsto -\sqrt{p_{+1}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_2\gamma_3\gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{-1}} \mapsto -\sqrt{p_{-1}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_2\gamma_3(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{+2}} \mapsto -\sqrt{p_{+2}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_1\gamma_3\gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$
- $\sqrt{p_{-2}} \mapsto -\sqrt{p_{-2}}$ corresponds to $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \mapsto \gamma_1\gamma_4(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w)$.

Hence, we see that the Monodromy group T_1 is isomorphic to Γ . \square

Next we want to calculate the Monodromy group of the 160 planes which intersect a surface in \mathcal{X} to give the 320 conics. First we will calculate the Monodromy group of the 16 planes associated to q_i for each $i \in [2, \dots, 10]$, which we denote each set by T_i . To do so, we will use the group Ω acting on $\mathbb{P}^3 \times \mathbb{P}^4$. This group permutes the 15 singular planes and the 10 points q_1, \dots, q_{10} as follows (using permutation notation):

- ϕ_1 acts as $(p_{+0}, p_{-0})(p_{+1}, p_{-1})(p_{+2}, p_{-2})(p_{+3}, p_{-3})$ and as $(q_5, q_6)(q_7, q_8)(q_9, q_{10})$,
- ϕ_2 acts as $(q_{+D}, q_{+E})(q_{-D}, q_{-E})(p_{+2}, p_{+3})(p_{-2}, p_{-3})$ and as $(q_1, q_2)(q_7, q_9)(q_8, q_{10})$,
- ϕ_3 acts as $(q_{+C}, q_{+D})(q_{-C}, q_{-D})(p_{+1}, p_{+2})(p_{-1}, p_{-2})$ and as $(q_2, q_3)(q_5, q_7)(q_6, q_8)$,
- ϕ_4 acts as $(q_{+D}, q_{-D})(q_{+E}, q_{-E})(p_{+0}, p_{-1})(p_{-0}, p_{+1})(p_{+2}, p_{-3})(p_{-2}, p_{+3})$ and as $(q_1, q_2)(q_3, q_4)(q_5, q_6)$,
- ϕ_5 acts as $(A, q_{+C})(q_{+D}, p_{+0})(q_{-D}, p_{-1})(q_{+E}, p_{-0})(q_{-E}, p_{+1})(p_{+2}, p_{-3})$ and as $(q_1, q_5)(q_2, q_6)(q_7, q_{10})$.

Hence we can find the Monodromy group of T_i from the Monodromy group of T_1 . Pick an element $\phi \in \Omega$ which permutes q_1 and q_i , then the element of the Monodromy group associated to the singular hyperplane H is $\phi^{-1} \cdot \gamma_{\phi(H)} \cdot \phi$, where $\gamma_{\phi(H)}$ is the element of the Monodromy group of T_1 . Note that since Γ is normal in Ω , we necessarily end up with an element of Γ .

Example. We work out explicitly some of the cases for the point q_2 . We use the element ϕ_2 which permutes the point q_1 and q_2 .

Since $\phi_2(A) = A$, the element corresponding to the hyperplane A in the Monodromy group of T_2 is $\phi_2^{-1} \gamma_A \phi_2 = \phi_2^{-1} \cdot \text{id} \cdot \phi_2 = \text{id}$.

Since $\phi_2(q_{+E}) = q_{+D}$, the element corresponding to the hyperplane q_{+E} in the Monodromy group of T_2 is $\phi_2^{-1} \gamma_{q_{+D}} \phi_2 = \phi_2^{-1} \cdot \gamma_4 \cdot \phi_2 = \gamma_3\gamma_4$.

We summarise the information in Table 2 below, the row headings are the 15 singular hyperplanes and the column headings are the 10 nodes q_i . Each entry is an element $\gamma \in \Gamma$ and represents how changing the sign of the square root of that singular hyperplane permutes the 16 planes associated to q_i (which we know can be represented as an element of Γ). An empty box stands for the identity element in Γ .

Theorem 4.5. *The Monodromy group of the 160 planes is C_2^9 .*

Proof. We use the information given in Table 2. The Monodromy group is a subgroup of S_{160} . It can be checked that the subgroup of S_{160} generated by A, \dots, p_{-3} (once they have been embedded in S_{160}) has order 2^9 . Now as each of the generators A, \dots, p_{-3} commutes with each other and have order 2, we know that every non-trivial element of the Monodromy group have order 2. Since the only group of order 2^9 with every non-trivial elements being involutions is C_2^9 , the Monodromy group of the 160 planes is C_2^9 . \square

Definition 4.6. Each conic comes in a natural pair, i.e., each plane intersecting the K3 surface gives two conics. We shall call two such conics *conjugates* of each other.

Recall from Page 3 that given the K3 surface X_p and the plane $T : r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$ associated to the point q_1 , then the two conics in $T \cap X_p$ are $Q_1 + \sqrt{\frac{\mu}{\Delta}}Q'$ and $Q_1 - \sqrt{\frac{\mu}{\Delta}}Q'$. Let $r_\mu = \sqrt{\frac{\mu}{\Delta}} \cdot a_2$, then the conics are expressed as $a_2Q_1 \pm r_\mu Q'$. Then, using the same method as in the proof of Lemma 4.1, we can express r_μ and a_2 explicitly in terms of A, B, C, D, E and find that:

$$r_\mu = \frac{\sqrt{-q+Cq+Dq-E}}{\sqrt{\Delta}} \left(b_1\sqrt{p_{-0}p_{+0}} + b_2\sqrt{p_{+1}p_{-1}} + b_3\sqrt{p_{+2}p_{-2}} + b_4\sqrt{p_{+0}p_{-0}p_{+1}p_{-1}p_{+2}p_{-2}} \right)$$

$$a_2 = b_5\sqrt{p_{+1}p_{-1}p_{+2}p_{-2}} + b_6\sqrt{p_{+0}p_{-0}p_{+2}p_{-2}} + b_7\sqrt{p_{+0}p_{-0}p_{+1}p_{-1}} + b_8$$

where $b_i \in \mathbb{Z}[A, B, C, D, E]$. From the equation of the conics $a_2Q_1 \pm r_\mu Q'$ we can get the equations of the other 30 conics defined by the point q_1 , using the action of Γ . Hence we see that to calculate the Monodromy group of the 32 conics defined by the point q_1 we need to look at loops going around the 15 singular hyperplanes and the Segre cubic. We first use the following lemma.

Lemma 4.7. *Let $|x| < \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$, then x satisfies $|ax + bx^2| \leq |ax| + |bx^2| < |c|$. In particular this implies that $c + ax + bx^2 \neq 0$.*

Proof. This is a simple cases by cases proof:

- Case 1.* $|x| < 1 = \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$. Then $|ax| + |bx^2| < |a| + |b|$, since $1 \leq |\frac{c}{2a}|, |\frac{c}{2b}|$ we know that $a, b \leq |\frac{c}{2}|$. Hence $|a| + |b| \leq |c|$.
- Case 2.* $|x| < \frac{1}{|a|} = \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$. Then $|ax| + |bx^2| < 1 + |\frac{b}{a^2}| \leq 1 + |\frac{b}{a}|$. Since $\frac{1}{|a|} \leq \frac{1}{|b|}$ implies $|\frac{b}{a}| \leq 1$, and $\frac{1}{|a|} \leq \frac{|c|}{|2a|}$ implies $2 \leq |c|$, then $1 + |\frac{b}{a}| \leq |c|$.
- Case 3.* $|x| < \frac{1}{|b|} = \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$. Then $|ax| + |bx^2| < |\frac{a}{b}| + |\frac{1}{b}|$. As in case 2, we see that $|\frac{a}{b}| \leq 1$ and $2 \leq |c|$, hence $|\frac{a}{b}| + |\frac{1}{b}| \leq 2 \leq |c|$.
- Case 4.* $|x| < |\frac{c}{2a}| = \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$. Then $|ax| + |bx^2| < |\frac{c}{2}| + |\frac{bc^2}{2a^2}| \leq |\frac{c}{2}| + |\frac{bc}{2a}|$. As $|\frac{c}{2a}| \leq |\frac{c}{2b}|$ implies $|\frac{b}{a}| \leq 1$, we have that $|\frac{c}{2}| + |\frac{b}{a}| |\frac{c}{2}| \leq |c|$.
- Case 5.* $|x| < |\frac{c}{2b}| = \min\{1, |\frac{c}{2a}|, |\frac{c}{2b}|, |\frac{1}{a}|, |\frac{1}{b}|\}$. Then $|ax| + |bx^2| < |\frac{a}{b}| |\frac{c}{2}| + |\frac{c}{2b}| |\frac{c}{2}| \leq |\frac{c}{2}| + |\frac{c}{2}| \leq |c|$.

\square

Proposition 4.8. *Given a non-singular K3 surface defined by the point $p = [A, B, C, D, E]$, we can find a loop based at p that changes the sign of $\sqrt{q+C}$ and leaves the sign of 15 square roots $\{\sqrt{-q-C}, \dots, \sqrt{p+3}, \sqrt{\Delta}\}$ unchanged.*

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
A					γ_3	γ_3	γ_4	γ_4	$\gamma_3\gamma_4$	$\gamma_3\gamma_4$
$q+C$	γ_3	γ_3					$\gamma_1\gamma_3$	γ_1	$\gamma_1\gamma_3$	γ_1
$-q-C$			γ_3	γ_3			$\gamma_1\gamma_4$	$\gamma_1\gamma_3\gamma_4$	$\gamma_1\gamma_3\gamma_4$	$\gamma_1\gamma_4$
$q+D$	γ_4		γ_4		$\gamma_2\gamma_4$	γ_2			$\gamma_2\gamma_4$	γ_2
$-q-D$		γ_4		γ_4	$\gamma_2\gamma_3$	$\gamma_2\gamma_3\gamma_4$			$\gamma_2\gamma_3\gamma_4$	$\gamma_2\gamma_3$
$q+E$		$\gamma_3\gamma_4$	$\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$		
$-q-E$	$\gamma_3\gamma_4$			$\gamma_3\gamma_4$	$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_4$	$\gamma_1\gamma_2\gamma_4$	$\gamma_1\gamma_2\gamma_3$		
$p+0$	$\gamma_1\gamma_2$	γ_2	γ_1		γ_1		γ_2		$\gamma_1\gamma_2$	
$p-0$	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_2\gamma_4$	$\gamma_1\gamma_3$			$\gamma_1\gamma_3$		$\gamma_2\gamma_4$		$\gamma_1\gamma_2\gamma_3\gamma_4$
$p+1$	$\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2\gamma_4$		$\gamma_1\gamma_3$	$\gamma_1\gamma_3$			$\gamma_2\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_4$
$p-1$	$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$		γ_1		γ_1	$\gamma_2\gamma_3$		$\gamma_1\gamma_2\gamma_3$	
$p+2$	$\gamma_1\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_4$		$\gamma_1\gamma_3\gamma_4$	$\gamma_2\gamma_4$			$\gamma_1\gamma_2\gamma_3$
$p-2$	$\gamma_1\gamma_4$		$\gamma_1\gamma_2\gamma_4$	γ_2	$\gamma_1\gamma_4$			γ_2	$\gamma_1\gamma_2\gamma_4$	
$p+3$		$\gamma_1\gamma_4$	$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3\gamma_4$		$\gamma_1\gamma_4$		$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3\gamma_4$	
$p-3$		$\gamma_1\gamma_3\gamma_4$	$\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3\gamma_4$		$\gamma_2\gamma_3\gamma_4$			$\gamma_1\gamma_2$

Table 2:

Proof. We construct the same loop as in Theorem 4.3 with a slight modification, this time we put the constraint that $\rho \in \mathbb{R}_{>0}$ satisfies

$$\rho < \min \left\{ |B + 2D + 2E|, |-B + 2D + 2E|, |8A + B + 2D - 2E|, |8A - B + 2D - 2E|, 1, \right. \\ \left. \frac{1}{|-4A^2 + 4DE|}, \frac{1}{|4A|}, \frac{|A^2B - 4A(D^2 + E^2) + 8ADE|}{2|-4A^2 + 4DE|}, \frac{|A^2B - 4A(D^2 + E^2) + 8ADE|}{2|4A|} \right\}.$$

The extra condition means that by Lemma 4.7 $\Delta = A^2B - 4A(D^2 + E^2) + 8ADE + (4CD - 4A^2)f(t) - 4Af(t)^2 \neq 0$ during the second segment of the loop. At the same time, we see that the sign of the square root cannot change. Finally, this extra condition on ρ does not effect the rest of the proof of Theorem 4.3. \square

We also prove that we can construct a loop based at p which goes around the Segre cubic but not the 15 singular hyperplanes.

Proposition 4.9. *Given a non-singular K3 surface defined by the point $p = [A, B, C, D, E]$, there exists a loop based at p that changes the sign of $\sqrt{\Delta}$ and leaves the sign of the 15 square roots $\{\sqrt{q+C}, \dots, \sqrt{p+3}\}$.*

Proof. First we claim that given such a K3 surface there exists B' such that the surface $[A, B', C, D, E]$ lies on the Segre cubic but not on the 15 singular hyperplanes. The first part of the statement is easy to see, just solve $\Delta = 0$ in terms of B' , which, since $A \neq 0$ as our surface is non-singular, has a solution as we are working over $\overline{\mathbb{Q}}$ (in fact B' is at worst in a degree 2 extension of the field of definition of A, B, C, D, E). For the second part, recall that if a point lies on the Segre cubic and one of the 15 singular hyperplanes, then it lies on a Segre plane, i.e., it must lie on a further two singular hyperplanes. But note that any surface lying on a Segre plane must lie on one singular hyperplane which is defined with no B (and hence B') terms. Therefore if $[A, B', C, D, E]$ lied on such an singular hyperplane, then so would $[A, B, C, D, E]$, contradicting our assumption that the surface is non-singular.

We will construct a loop in the same way as in proof of Theorem 4.3. Note that B can be written uniquely as $B' + re^{i\phi}$ for some $r \in \mathbb{R}_{>0}$ and $\phi \in [0, 2\pi)$. Define a loop $\tilde{\gamma}(t) = [\tilde{\gamma}_A(t), \tilde{\gamma}_B(t), \tilde{\gamma}_C(t), \tilde{\gamma}_D(t), \tilde{\gamma}_E(t)]$ ($0 \leq t \leq 3$) as $\tilde{\gamma}_B = iB' + f(t)$ and $\tilde{\gamma}_i = i$ for all t and $i \in \{A, B, D, E\}$, where f is composed of the following 3 segments:

$$f(t) = \begin{cases} (\rho t + r(1-t))e^{i\phi} & t \in [0, 1] \\ \rho e^{i\phi + i(t-1)2\pi} & t \in [1, 2] \\ (\rho(3-t) + r(t-2))e^{i\phi} & t \in [2, 3] \end{cases}$$

and $\rho \in \mathbb{R}_{>0}$ satisfies

$$\rho < \min\{|4A \pm B' + 2C + 2D + 2E|, |4A \pm B' + 2C - 2D - 2E|, \\ |4A \pm B' - 2C + 2D - 2E|, |4A \pm B' - 2C - 2D + 2E|, |2B'|\}.$$

Note that with the conditions on ρ the path never loops around the 15 singular hyperplanes, hence as we have seen before, we do not have a sign change from them. As for $\sqrt{\Delta}$, note that the first and third segments leave it untouched, while for the second segment

$$\begin{aligned} \Delta &= 16A^3 - 4A(C^2 + D^2 + E^2) + 4CDE + A(iB' + \rho e^{i\phi + i(t-1)e\pi})^2 \\ &= -AB'^2 + A(B' + \rho e^{i\phi + i(t-1)e\pi})^2 \\ &= A(B' + \rho e^{i\phi + i(t-1)e\pi} + B') (B' + \rho e^{i\phi + i(t-1)e\pi} - B') \\ &= A\rho e^{i\phi + i(t-1)e\pi} (2B' + \rho e^{i\phi + i(t-1)e\pi}). \end{aligned}$$

Hence as in the previous proof, we find that as we loop around $\Delta = 0$, the sign of $\sqrt{\Delta}$ changes.

Note that the above only works under the assumption that $B' \neq 0$. In the case $B' = 0$, we first need to find a path from our point $[A, B, C, D, E]$ to the point $[A + \epsilon, B, C, D, E]$, where ϵ is small enough that we do not go near any singular hyperplane or the Segre cubic. In that case, we use the point $[A + \epsilon, B, C, D, E]$ as our starting point. \square

Hence we can use the explicit equations of the conics to find the Monodromy group of the 32 conics defined by the point q_1 . Then, as before, we use the group Ω acting on our set of points $\{q_i\}$ and 15 singular hyperplanes to find the Monodromy group of the 32 conics defined by each of the points q_i . We summarized the information in Table 3, where again the rows are the singular hyperplanes of the Segre cubic we looped around and the columns are the points q_i . The entries are either elements of Γ , or -1 which denotes the element that conjugates conics, that is permutes conics defined on the same plane.

Theorem 4.10. *The Monodromy group of the 320 conics is C_2^{10} .*

Proof. We use the information given in Table 3. The Monodromy group is a subgroup of S_{320} . It can be checked that the subgroup of S_{320} generated by Δ, \dots, p_{-3} (once they have been embedded in S_{320}) has order 2^{10} . Now as each of the generators Δ, \dots, p_{-3} commutes with each other and have order 2, we know that every non-trivial element of the Monodromy group have order 2. Since the only group of order 2^{10} with every non-trivial elements being involutions is C_2^{10} , the Monodromy group of the 320 conics is C_2^{10} . \square

Corollary 4.11. *The moduli space of pairs (X, C) , where X is an Heisenberg-invariant quartic K3 surfaces and C a conic on X , has 10 irreducible components.*

Proof. Let Z be the moduli space of pairs (X, C) with X a surface in \mathcal{X} and C a conic lying on X . We showed that the Monodromy group of $\pi, \pi : Z \rightarrow \mathbb{P}_{[A,B,C,D,E]}^4$, breaks the 320 conics on X in 10 orbits, each orbit having size 32. Since calculating the Monodromy group involves lifting a path in $\mathbb{P}_{[A,B,C,D,E]}^4$ to a path in Z , any two elements in the same orbit represent two connected elements in Z . Finally, since the paths avoided where π was not smooth, the 10 orbits correspond to 10 smooth connected components of Z , i.e. 10 irreducible components. \square

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	β_1	β_2	β_3	β_4	β_5	β_6	β_7	β_8	β_9	β_{10}
Δ	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
A					$-\gamma_3$	$-\gamma_3$	$-\gamma_4$	$-\gamma_4$	$-\gamma_3\gamma_4$	$-\gamma_3\gamma_4$
q_{+C}	$-\gamma_3$	$-\gamma_3$					$-\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_1$
$-q_{-C}$			$-\gamma_3$	$-\gamma_3$			$-\gamma_1\gamma_4$	$-\gamma_1\gamma_3\gamma_4$	$-\gamma_1\gamma_3\gamma_4$	$-\gamma_1\gamma_4$
q_{+D}	$-\gamma_4$		$-\gamma_4$		$-\gamma_2\gamma_4$	$-\gamma_2$			$-\gamma_2\gamma_4$	$-\gamma_2$
$-q_{-D}$		$-\gamma_4$		$-\gamma_4$	$-\gamma_2\gamma_3$	$-\gamma_2\gamma_3\gamma_4$			$-\gamma_2\gamma_3\gamma_4$	$-\gamma_2\gamma_3$
q_{+E}		$-\gamma_3\gamma_4$	$-\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$	$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$		
$-q_{-E}$	$-\gamma_3\gamma_4$			$-\gamma_3\gamma_4$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_4$	$-\gamma_1\gamma_2\gamma_4$	$-\gamma_1\gamma_2\gamma_3$		
p_{+0}	$-\gamma_1\gamma_2$	$-\gamma_2$	$-\gamma_1$		$-\gamma_1$		$-\gamma_2$		$-\gamma_1\gamma_2$	
p_{-0}	$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_2\gamma_4$	$-\gamma_1\gamma_3$			$-\gamma_1\gamma_3$		$-\gamma_2\gamma_4$		$-\gamma_1\gamma_2\gamma_3\gamma_4$
p_{+1}	$-\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2\gamma_4$		$-\gamma_1\gamma_3$	$-\gamma_1\gamma_3$			$-\gamma_2\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_4$
p_{-1}	$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$		$-\gamma_1$		$-\gamma_1$	$-\gamma_2\gamma_3$		$-\gamma_1\gamma_2\gamma_3$	
p_{+2}	$-\gamma_1\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_4$		$-\gamma_1\gamma_3\gamma_4$	$-\gamma_2\gamma_4$			$-\gamma_1\gamma_2\gamma_3$
p_{-2}	$-\gamma_1\gamma_4$		$-\gamma_1\gamma_2\gamma_4$	$-\gamma_2$	$-\gamma_1\gamma_4$			$-\gamma_2$	$-\gamma_1\gamma_2\gamma_4$	
p_{+3}		$-\gamma_1\gamma_4$	$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3\gamma_4$		$-\gamma_1\gamma_4$		$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3\gamma_4$	
p_{-3}		$-\gamma_1\gamma_3\gamma_4$	$-\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$	$-\gamma_1\gamma_3\gamma_4$		$-\gamma_2\gamma_3\gamma_4$			$-\gamma_1\gamma_2$

Table 3: